A Novel Static PET Image Reconstruction Method

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***Abstract*—In this paper, we present a novel image reconstruction algorithm for positron emission tomography(PET). Almost all of existing reconstruction approaches assume that the measurement model for PET is linear equation with Gaussian white noise or energy-bounded noise, which only approximates the emission and detection of PET very roughly. In fact, the real situation is much more complicated than the one mentioned above and there must be something that is not be involved in the aforementioned model. Hence, in this paper, we establish a more general and vivid measurement model via involving an unknown input, and propose a reconstruction method based on the optimal filtering for the stochastic system with unknown input. The approach reconstructs the PET image effectively and its performance is evaluated with the computer-synthesized cardiac-phantom.**

***Index Terms*—optimal filter, PET, image reconstruction, stochastic system, unknown input, convergence, stability**

I. Introduction

Positron Emission Tomography (PET) is a medical imaging technology based on nuclear physics and molecular biology. It has been widely used in the diagnosis of cancer, cardiac disease, neurological and psychiatric diseases and drug filtrate and development. Unlike CT imaging, PET is a kind of functional imaging technique that can detects the metabolism of organisms, reflect the physiological or pathological changes of the organism at the molecular level, which can provide effective evidence for the detection and diagnosis of early diseases. In the PET scan, the medical cyclotron is used to generate the positron emission nuclide, and then the tracer labeled by radioactive isotope is injected into the organism. Through the blood circulation, the certain amount of tracer will appear in the tissues and organs of the organism. The radioactive isotope nuclides are extremely unstable and they will decay. In the process of decay, positrons are produced and they will annihilate with free electrons in the surrounding tissues and organs to produce a pair of gamma-like photons with almost equal energies. PET scanning can capture these photons and generate projection data. Based on these projection data and reconstruction algorithms, the concentration distribution of the tracer in tissues and organs can be reconstructed. [1]

The paper mainly considers the static PET image reconstruction methods, they mainly can be divided into two categories: analytic methods [2] and iterative methods [3]. The filtered back projection (FBP) method [4] is the typical analytic method. Although its computational cost is lower, it always suppresses noise not well, which results in a poor quality of the reconstructed image [5] [6] [7]. Due to good data adaptability, statistical iterative methods such as ML-EM, SOR and PWLS [8] [9], can provide clearer reconstruction image than analytic methods and thus become popular. The introduction of state space model makes it possible to reconstruct PET images based on Kalman filtering [10], H_∞ filtering [11] and particle filtering [12] [13] [14]. However, the measurement model for the filtering reconstruction method is assumed to be linear with Gaussian white noise or energy-bounded noise approximately. Gaussian-white-noise model fails to describe those not modeled and energy-bounded noise model is too conservative because it takes all the information except for the linear part as the energy-bounded only and neglect information that we know better. For instance, the imaging systems in PET are subject to a range of effects. Some effects we can describe well, but some effects we can not describe Toward this end, we have established a more general and vivid measurement model by involving an unknown input to represent those we can not describe [15] [16].

The organization of the paper is as follows. The PET image reconstruction problem is reformulated in the following section. The solution to the problem is derived in Section3. Numerical experiments are given in Section 4. Some conclusion remarks are achieved in Section 5.

II. Problem Reformulation

PET image reconstruction generally uses an reconstruction approach to recover the concentration distribution of the trace based on the sinogram. This sinogram records positrons annihilations captured by the coincident detection. It connected with the concentration distribution of the trace via the relationship as follows [17] [18]

$$y_t = D_t x_t + v_t \tag{1}$$

where $y_t = col\{y_{t,i}|i = 1, \ldots\ldots M\}$ represents the observed sinogram data at instant t and $y_{t,i}$ means the total coincidences of the ith detector bin at instant t and M the total number of detector bins. $x_t = col\{x_{t,j}|j = 1, \ldots\ldots N\}$ represents the concentration distribution of the trace at instant t. $x_{t,j}$ indicates the concentration distribution of the trace in voxel j at instant t and N is the total number of voxels. D_t represents the projection matrix and reflects the projection relationship between the concentration distribution of the trace and the sinogram data in the human body at instant t. Generally speakingto facilitate discussion , the observation noise v_t is assumed to be mutually independent Gaussian random vectors with zero mean and variance matrices R_t, respectively, namely,

$$R_t = E\left[v_t v_t^T\right] > 0 \tag{2}$$

From the previous analysis, the measurement equation (1)is not enough to describe the relationship between the concentration distribution of the trace and the sonogram. Therefore, in this paper, we introduce an unknown input d_t in (1) at instant t , so that the measurement (1) can be modified as

$$y_t = D_t x_t + H_t d_t + v_t \tag{3}$$

where H_t is a given matrix.

To simplify statement, we only consider corresponding static PET image reconstruction, which means evolution of the concentration distribution of the trace of (3) can be described as follows [19]:

$$x_{t+1} = x_t \tag{4}$$

Therefore, the image reconstruction of PET can be reformulated as the estimation problem below as

$$\min_{\hat{x}_t} E\left\|x_t - \hat{x}_t\right\| \tag{5}$$

$$s.t.(3)and(4) \tag{6}$$

where $\hat{x}_t$ stands for an estimation for x_t in terms of all information available until instant t. $\|\|$ represents 2-norm in Euclidean Space.

III. Reconstruction Results

After an analysis over measurement model, the PET reconstruction problem is restated as an optimal filtering as (5) and (6) in the last section. What follows is to derive the solution to the problem.

In order to facilitate the following statement, some preliminary work will be provided as follows.

Make singular value decomposition for H_t then

$$H_t = U_t \begin{bmatrix} \Sigma_t & 0 \\ 0 & 0 \end{bmatrix} V_t^T = \left[U_{1,t}\ U_{2,t}\right] \begin{bmatrix} \Sigma_t & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1,t}^T \\ V_{2,t}^T \end{bmatrix} \tag{7}$$

where Σ_t is a diagonal matrix of full rank, while $U_t = [U_{1,t}\ U_{2,t}]$ and $V_t = [V_{1,t}\ V_{2,t}]$ are unitary matrices.

Next, according to the decomposition of H_t, we decouple the sinogram data of the observation equation (3) by a nonsingular transformation

$$T_t = \left(\begin{bmatrix} I_{n_{d_{1,t}} \times n_{d_{1,t}}} & -U_{1,t}^T R_t U_{2,t} (U_{2,t}^T R_t U_{2,t})^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} U_{1,t}^T \\ U_{2,t}^T \end{bmatrix} \right) \tag{8}$$

where $n_{d_{1,t}}$ is the rank of H_t .Let $T_t = \left[\begin{array}{cc} T_{1,t}^T & T_{2,t}^T \end{array}\right]$, $T_{1,t}$ and $U_{1,t}$ have the same dimension, so do$T_{2,t}$ and $U_{2,t}$ are the same.

Then the observation equation (3) was decoupled two observation equation $z_{1,t}$ and $z_{2,t}$ as follows.

$$\begin{aligned} z_{1,t} = T_{1,t} y_t = D_{1,t} x_t + \Sigma_t d_{1,t} + v_{1,t} \\ z_{2,t} = T_{2,t} y_t = D_{2,t} x_t + v_{2,t} \end{aligned} \tag{9}$$

Associate (3) with (8)

$$D_{1,t} = T_{1,t} D_t, D_{2,t} = T_{2,t} D_t \tag{10}$$

$$v_{1,t} = T_{1,t} v_t, v_{2,t} = T_{2,t} v_t \tag{11}$$

$$\begin{aligned} d_t = \left[\begin{array}{cc} V_{1,t} & V_{2,t} \end{array}\right] \begin{bmatrix} d_{1,t} \\ d_{2,t} \end{bmatrix} = V_{1,t} d_{1,t} + V_{2,t} d_{2,t} \\ d_{1,t} = V_{1,t}^T d_t, d_{2,t} = V_{2,t}^T d_t \end{aligned} \tag{12}$$

This transform is also chosen such that the decoupled measurement noise term is uncorrelated. The covariances of $v_{1,t}$ and $v_{2,t}$.

$$\begin{aligned} R_{1,t} = E\left[v_{1,t} v_{1,t}^T\right] = T_{1,t} R_t T_{1,T}^T > 0 \\ R_{2,t} = E\left[v_{2,t} v_{2,t}^T\right] = U_{2,T}^T R_t U_{2,t} = T_{2,t} R_t T_{2,T}^T > 0 \\ R_{12,t} = E\left[v_{1,t} v_{2,t}^T\right] = T_{1,t} R_t T_{2,T}^T > 0 \\ R_{12,(t,i)} = E\left[v_{1,t} v_{2,i}^T\right] = T_{1,t} E\left[v_t v_i^T\right] T_{2,i}^T = 0 \quad \forall t \neq i \end{aligned} \tag{13}$$

Because measurement noise is independent of the initial state, and it is Gaussian white, we have

$$\begin{aligned} \text{cov}\left[v_{1,t}, v_{1,i}\right] = 0 \quad \forall t \neq i \\ \text{cov}\left[v_{2,t}, v_{2,i}\right] = 0 \quad \forall t \neq i \\ \text{cov}\left[v_{1,t}, x_0\right] = T_{1,t} \text{cov}\left[v_t, x_0\right] = 0 \\ \text{cov}\left[v_{2,t}, x_0\right] = T_{2,t} \text{cov}\left[v_t, x_0\right] = 0 \end{aligned} \tag{14}$$

Denote $\hat{x}_t$ as the estimation of x_t. Let P_t be the estimation error covariance matrix of the concentration distribution of the tracer, namely,

$$P_t = E\left[\left(x_t - \hat{x}_t\right)\left(x_t - \hat{x}_t\right)^T\right] \tag{15}$$

Now we are in the position to give the following reconstruction algorithm.

Theorem 1: Given the measurement (3) and system (4), and then given measurements up to instant $t-1$, the optimal linear filter $\hat{x}_t$ in the minimum-variance unbiased sense can be summarized as follows

$$\hat{x}_t = \hat{x}_{t-1} + L_t \tilde{z}_{2,t} \tag{16}$$

$$\tilde{z}_{2,t} = z_{2,t} - D_{2,t} \hat{x}_{t-1} \tag{17}$$

$$L_t = P_{t-1} D_{2,t}^T \left(D_{2,t} P_{t-1} D_{2,t}^T + R_{2,t}\right)^{-1} \tag{18}$$

$$P_t = (I - L_t D_{2,t}) P_{t-1} (I - L_t D_{2,t})^T + L_t R_{2,t} {L_t}^T \tag{19}$$

Proof 1: Before calculating the filtering gain L_t, we start with proving the unbiasedness of the estimation $\hat{x}_t$. Assume $\hat{x}_{t-1}$ is the unbiased estimation of x_{t-1}, which means

$$E\left[e_{t-1}\right] = E\left[x_{t-1} - \hat{x}_{t-1}\right] = 0 \tag{20}$$

In terms of (4), (16), and (17). It is not hard to derive

$$\begin{aligned} E\left[e_t\right] &= E\left[x_t - \hat{x}_t\right] \\ &= E\left[x_t - \hat{x}_{t-1} + L_t \hat{z}_{2,t}\right] \\ &= x_{t-1} - \hat{x}_{t-1} + L_t\left(D_{2,t} x_t + v_{2,t} - D_{2,t}\hat{x}_{t-1}\right) \\ &= \left(I - L_t D_{2,t}\right) e_{t-1} + L_t v_{2,t} \end{aligned} \tag{21}$$

Due to the assumption that $\hat{x}_{t-1}$ is the unbiased estimation of x_{t-1} and $v_{2,t}$ is white noise with zero mean, we have

$$E\left[e_t\right] = E\left[x_t - \hat{x}_t\right] = 0 \tag{22}$$

Hence, $\hat{x}_t$ is unbiased. As a matter of fact, (21) shows that e_t can be represented by the combination of x_0, $v_{1,0}$,$v_{1,t-1}$, $v_{2,0}$,,$v_{2,t}$, If $\hat{x}_0 = E\left[x_0\right]$, it is obvious that $E\left[e_t\right] = E\left[x_t - \hat{x}_t\right] = 0$.

Now we turn to find the filtering gain L_t so that the filter (16) is optimal linear in the minimum-variance unbiased sense.

By virtue of (4) (15) (16) and (17), the estimation error covariance matrix P_t adimit

$$\begin{aligned} P_t &= E\left[e_t e_t^T\right] \\ &= E\{\left[\left(I - L_t D_{2,t}\right) e_{t-1} \right. \\ &\left. - L_t v_{2,t}\right]\left[\left(I - L_t D_{2,t}\right) e_{t-1} - L_t v_{2,t}\right]^T\} \\ &= \left(I - L_t D_{2,t}\right) E\left[e_{t-1} e_{t-1}^T\right]\left(I - L_t D_{2,t}\right)^T \\ &- L_t E\left[v_{2,t} e_{t-1}^T\right]\left(I - L_t D_{2,t}\right)^T \\ &- \left(I - L_t D_{2,t}\right) E\left[e_{t-1} v_{2,t}^T\right] L_t^T \\ &= \left(I - L_t D_{2,t}\right) P_{t-1}\left(I - L_t D_{2,t}\right)^T + L_t R_{2,t} L_t^T \end{aligned} \tag{23}$$

The last equality in (23) originates from (14) and (21).

Let $J_t = trP_t$, then

$$\frac{\partial J_t}{\partial L_t} = 2\left(I - L_t D_{2,t}\right) P_{t-1}\left(-D_{2,t}^T\right) + 2 L_t R_{2,t} \tag{24}$$

gives

$$L_t R_{2,t} = \left(I - L_t D_{2,t}\right) P_{t-1} D_{2,t} \tag{25}$$

Because R_t and T_t are nonsingular, so is $R_{2,t}$. Therefore,

$$L_t = P_{t-1} D_{2,t}\left(D_{2,t} P_{t-1} D_{2,t}^T + R_{2,t}\right)^{-1} \tag{26}$$

(18) is proved.

A recursive algorithm for image reconstruction is provided as follows:

1.Initialize P_0 and $\hat{x}_0$;

2.Compute the gain matrix L_t via (18);

3.Update the filter, the estimation error of the measurement, the estimation error covariance matrix of the state by (16), (17), (19), respectively;

4.Repeat 2 and 3 until obtaining a satisfying filter.

Remark 1: In this paper, the algorithm essentially rejects the observation $z_{1,t}$ which contain the unknown input, using only the observation $z_{2,t}$ without the unknown input.

IV. Numerical Experiments

The reconstruction method in the paper is validated with the computer-synthesized Cardiac-phantom. The digital Cardiac-phantom and its sinogram are shown in Fig. 1 and Fig. 2, respectively. The data set used for validation is generated by Monte Carlo simulations.

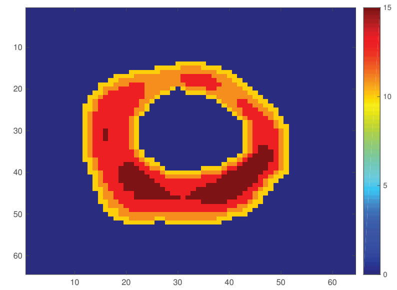

Fig. 1. Digital phantom generated from cardiac thorax

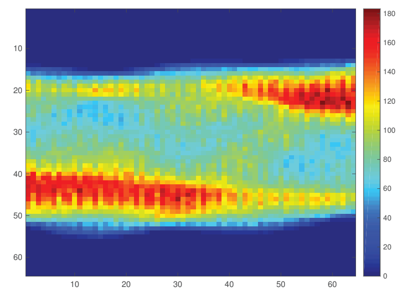

Fig. 2. Sinogram

We compare the reconstruction performance of our algorithm (NF) with that of the algorithm based on the Kalman filtering (KF). In detail, we consider different level of disturbance induced by the unknown input. Fig. 3 and Fig. 4 show that when only 5 of 4096 measurements actually involve the unknown inputs, both algorithms reconstruct the image well although the differences is not so clear. Fig. 5 and Fig. 6 show that when 1024 of 4096 measurements involve the unknown inputs, NF still reconstructs the image well while KF generates a terrible result. In fact, it is not difficult to find that more measurement disturbed by the unknown inputs, the worse reconstruction results both algorithms yield. However, NF can always provide better reconstruction image in the same noise situation because it is proposed based on a more vivid measurement model.

From the Fig. 3, Fig. 4, Fig. 5, Fig. 6 and TABLE. I, we could find that the performance in our method is better than the KF under the unmatched noise situations.

V. Conclusion

This paper provides a new algorithm of image reconstruction for PET systems by introducing an unknown input in the

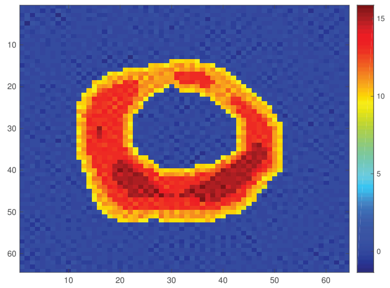

Fig. 3. KF at matched noise level

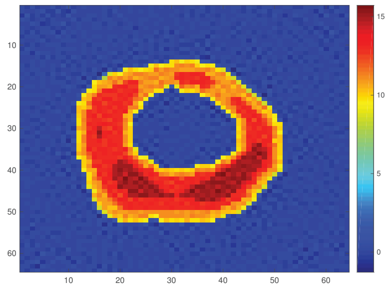

Fig. 4. NF at matched noise level

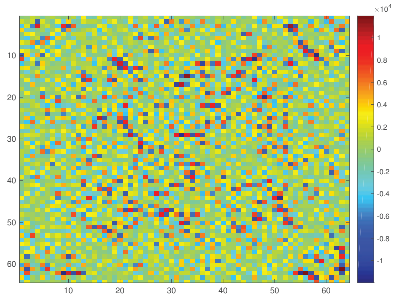

Fig. 5. KF at unmatched noise level

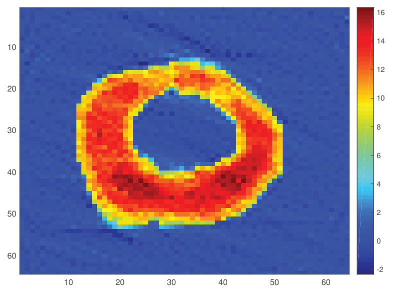

Fig. 6. NF at unmatched noise level

TABLE I
BIAS AND VARIANCE OF EACH METHOD

dimension of the involving unknown input	noise parameter	algorithm	bias± variance
5	matched noise	KF	1.5019e-05± 0.0018
		DF	-1.5650e-05 ± 0.0018
1024	unmatched noise	KF	2.2505±17.1729
		DF	-2.6647e-04±0.0046

measurement model of the PET. The algorithm outperforms the one based on the standard Kalman filtering when a linear measurement model only with Gaussian noise can not characterize the real situation of the emission and detection process well. The involvement of the unknown input divides the measurement into two parts, one with unknown input and the other without unknown input. Our algorithm is actually designed based on the measurement without unknown input, which eliminates the negative effects aroused by the unknown input. Simulation results have shown that the proposed approach reconstructs the PET image well.

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